First And Second Derivative Tests

1. First Derivative Test for Local Extrema:
   If $U \subset \mathbb{R}^n$ is open, the function $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is diff and $\vec{x_0} \in U$ is a local ext, then $Df(\vec{x_0}) = 0$.
   I.e. $\vec{x_0}$ is a critical point of f.

   Note: A critical point that is NOT a local ext is a saddle point.

2. Second Derivative Test for Local Extrema:
   Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ be of class C^3, $x_0 \in$ an open disk $\subset U$ be a crit point of f.

   Let $Hf(x_0)$ be the Hessian Matrix at x_0.

   If $Hf(x_0)$ is:
   1. Pos Def, then $(x_0, f(x_0))$ is a local min of f.
   2. Neg Def, then $(x_0, f(x_0))$ is a local max of f.
   3. Neither, but $\det(Hf(x_0)) \neq 0$, then $(x_0, f(x_0))$ is a saddle point.

   Note: If $\det(Hf(x_0)) = 0$, then it is of degenerate type.

3. How to Find if $Hf(x_0)$ is Pos Def, Neg Def, or Neither:

1. Eigenvalue Test:
We know that all Hessian Matrices are symmetric because of the thm of equality of mixed partials. Because all Hessian Matrices are symmetric, we are allowed to use the eigenvalue test as it only applies for symmetric matrices.

The test states:
If A is a symmetric matrix, then:
1. $\vec{x}^T A \vec{x}$ is pos def iff all eigenvalues of A are positive.
2. $\vec{x}^T A \vec{x}$ is neg def iff all eigenvalues of A are negative.
3. $\vec{x}^T A \vec{x}$ is indef iff A has at least one positive eigenvalue and at least one negative eigenvalue.

To see if $Hf(x_0)$ is pos def, neg def or neither, find all of its eigenvalues and use the test above.

2. Determinant Test:

Since all Hessian Matrices are symmetric, we may use the determinant test.

The determinant test states:
If A is a symmetric matrix, then
1. A is pos def iff the det of A_k, every $k \times k$ submatrix, is positive. I.e. $+ + + \ldots$
2. A is neg def iff the det of A_k alternates between negative and positive, starting with negative first. I.e. $- + - + - + \ldots$

E.g.

$$\begin{array}{c} \begin{array}{cc} A_1 & A_2 \end{array} \\ A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{array}$$

$|A_1| = 2 > 0$

$|A_2| = (2)(2) - (-1)(-1)$

$= 4 - 1$

$= 3 > 0$

$\therefore$ A is pos def.

$$\begin{array}{c} \begin{array}{cc} A_1 & A_2 \end{array} \\ B = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \end{array}$$

$|A_1| = -2 < 0$

$|A_2| = 3 > 0$

$\therefore$ A is neg def.

4. Second Derivative Test for Functions with 2 Variables:

Let $f(x,y)$ be of class C^2, $(x_0, y_0) \in$ an open disk $\subset U$ be a crit point of f satisfying $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.
Let $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$

If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local min at (x_0, y_0).

If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local max at (x_0, y_0).

If $D(x_0, y_0) < 0$, then f has a saddle point at (x_0, y_0).

If $D(x_0, y_0) = 0$, then the test is inconclusive.

5. Example: Find the critical points of $f(x,y) = xy(x-2)(y+3)$ and classify them.

Soln:

1. Find all critical points.

$$\frac{\partial f}{\partial x} = (y+3)\left[y(x-2) + xy\right] = 0$$

$\rightarrow (y+3)(2y)(x-1) = 0$

$\rightarrow (y+3)(y)(x-1) = 0$

$y = -3$ or $y = 0$ or $x = 1$

$$\frac{\partial f}{\partial y} = (x-2)\left[x(y+3) + xy\right] = 0$$

$\rightarrow (x)(x-2)(2y+3) = 0$

When $y = -3$, $x = 0$ or $x = 2$

$(0,-3)$ and $(2,-3)$ are 2 crit points.

When $y = 0$, $x = 0$ or $x = 2$

$(0,0)$ and $(2,0)$ are 2 crit points.

when $x=1$, $y=-\frac{3}{2}$
$(1,-\frac{3}{2})$ is a critical point.

In total, we have 5 crit points:
$(0,0)$, $(0,-3)$, $(1,-\frac{3}{2})$, $(2,-3)$ and $(2,0)$

To classify them, we can use the 2nd derivative test.

$$Hf = \begin{vmatrix} 2y(y+3) & (2x-2)(2y+3) \\ (2x-2)(2y+3) & 2x(x-2) \end{vmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} A_1 & A_2 \end{array} \\ Hf(0,0) = & \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} \end{array}$$

$\det(A_1)=0$
$\det(A_2)=-36$
Since it's neither pos def or neg def and $\det(A_2)\neq 0$, $(0,0)$ is a saddle point.

$$\begin{array}{cc} & \begin{array}{cc} A_1 & A_2 \end{array} \\ Hf(2,0) = & \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix} \end{array}$$

$\det(A_1)=0$
$\det(A_2)=-36$
$\therefore$ $(2,0)$ is a saddle point.

$$Hf(0,-3) = \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix}$$

$\therefore (0,-3)$ is a saddle point.

$$Hf(2,-3) = \begin{vmatrix} 0 & 6 \\ 6 & 0 \end{vmatrix}$$

$\therefore (2,-3)$ is a saddle point.

$$Hf(1,-\tfrac{3}{2}) = \begin{array}{c} \begin{array}{cc} A_1 & A_2 \end{array} \\ \begin{vmatrix} -\frac{9}{2} & 0 \\ 0 & -2 \end{vmatrix} \end{array}$$

$\det(A_1) = -\frac{9}{2}$

$\det(A_2) = 9$

Since this has the form $-\,+$, $Hf(1,-\frac{3}{2})$ is neg def. This means that $(1,-\frac{3}{2})$ is a local max.